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2004 J. Phys. A: Math. Gen. 37 5449

(http://iopscience.iop.org/0305-4470/37/20/013)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 5449–5466 PII: S0305-4470(04)73973-6

The nature of the essential spectrum in curved quantum waveguides

David Krejčiřík[1,3] **and Rafael Tiedra de Aldecoa**[2]

[1] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

[2] Département de Physique Théorique, Université de Genève, 24, quai E. Ansermet, 1211 Genève 4, Switzerland

E-mail: dkrej@math.ist.utl.pt and rafael.tiedra@physics.unige.ch

Received 5 January 2004, in final form 29 March 2004
Published 5 May 2004
Online at stacks.iop.org/JPhysA/37/5449
DOI: 10.1088/0305-4470/37/20/013

Abstract

We study the nature of the essential spectrum of the Dirichlet Laplacian in tubes about infinite curves embedded in Euclidean spaces. Under suitable assumptions about the decay of curvatures at infinity, we prove the absence of singular continuous spectrum and state properties of possible embedded eigenvalues. The argument is based on the Mourre conjugate operator method developed for acoustic multistratified domains by Benbernou (1998 *J. Math. Anal. Appl.* **225** 440–60) and Dermenjian *et al* (1998 *Commun. Partial Differ. Equ.* **23** 141–69). As a technical preliminary, we carry out a spectral analysis for Schrödinger-type operators in straight Dirichlet tubes. We also apply the result to the strips embedded in abstract surfaces.

PACS numbers: 03.65.Nk, 02.30.Tb
Mathematics Subject Classification: 81Q10, 35Q40, 58J50

1. Introduction

A strong physical motivation to study the Dirichlet Laplacian in infinitely stretched tubular regions comes from the fact that it constitutes a reasonable model for the Hamiltonian of a non-relativistic quantum particle in mesoscopic systems called *quantum waveguides* [11, 19, 26]. Since there exists a close relation between spectral and scattering properties of Hamiltonians, one is naturally interested in carrying out the spectral analysis of the Laplacian in order to understand the quantum dynamics in waveguides. For instance, the crucial step in most proofs of asymptotic completeness is to show that the Hamiltonian has no singular continuous spectrum [28]. The Laplacian in a tube has attracted considerable attention since

[3] On leave of absence from Department of Theoretical Physics, Nuclear Physics Institute, Academy of Sciences, 250 68 Řež, near Prague, Czech Republic.

it was shown in [15] that there may be discrete eigenvalues in curved waveguides. However, a detailed analysis of the essential part of the spectrum has been left aside up to now. The purpose of the present paper is to fill in this gap.

The usual model for a curved quantum waveguide, which we adopt in this paper, is as follows. Let $s \mapsto p(s)$ be an infinite unit-speed smooth curve in $\mathbb{R}^d$, $d \geqslant 2$ (the physical cases corresponding to $d = 2, 3$). Assuming that the curve possesses an appropriate smooth Frenet frame $\{e_1, \ldots, e_d\}$ (cf assumption 3.1), the ith curvature κ_i of p, $i \in \{1, \ldots, d-1\}$, is a smooth function of the arc-length parameter $s \in \mathbb{R}$. Given a bounded open connected set ω in $\mathbb{R}^{d-1}$ with the centre of mass at the origin, we identify the configuration space Γ of the waveguide with a tube of cross-section ω about p, namely,

$$\Gamma := \mathscr{L}(\mathbb{R} \times \omega) \qquad \mathscr{L}(s, u^2, \ldots, u^d) := p(s) + u^\mu \mathcal{R}_\mu{}^\nu(s) e_\nu(s) \tag{1.1}$$

where μ, ν are summation indices taking values in $\{2, \ldots, d\}$ and $(\mathcal{R}_\mu{}^\nu)$ is a family of rotation matrices in $\mathbb{R}^{d-1}$. In this paper, we choose the rotations in such a way that (s, u), with $u := (u^2, \ldots, u^d)$, are orthogonal 'coordinates' (cf section 3.1.3) due to the technical simplicity. It should be stressed here that while the shape of the tube Γ is not influenced by a special choice of $(\mathcal{R}_\mu{}^\nu)$ provided ω is circular, this may no longer be true for a general cross-section. We make the hypotheses (assumption 3.2) that κ_1 is bounded, $a\|\kappa_1\|_\infty < 1$, with $a := \sup_{u\in\omega} |u|$, and Γ does not overlap itself so that the tube can be globally parametrized by (s, u). Our object of interest is the Dirichlet Laplacian associated with the tube, i.e.

$$-\Delta_{\mathrm{D}}^\Gamma \quad \text{on} \quad L^2(\Gamma). \tag{1.2}$$

If p is a straight line, i.e. all $\kappa_i = 0$, then Γ may be identified with the straight tube $\Omega := \mathbb{R} \times \omega$. In that case, it is easy to see that the spectrum of (1.2) is purely *absolutely continuous* and equal to the interval $[\nu_1, \infty)$, where ν_1 denotes the first eigenvalue of the Dirichlet Laplacian in the cross-section ω.

On the other hand, if p is non-trivially curved and straight asymptotically, in the sense that the curvature κ_1 vanishes at infinity, then the essential spectrum of (1.2) remains equal to $[\nu_1, \infty)$. However, there are always *discrete eigenvalues* below ν_1. When $d = 2$, the latter was proved for the first time in [15] for a rapidly decaying curvature and sufficiently small a. Numerous subsequent studies improved and generalized this initial result [8, 11, 17, 23, 24, 30]. The generalization to tubes of circular cross-section in $\mathbb{R}^3$ was done in [17] (see also [11]) and the case of any dimension $d \geqslant 2$ and arbitrary cross-section can be found in [8]. Let us also mention that the discrete spectrum may be generated by other local perturbations of the straight tube Ω (see, e.g., [4, 7, 16]), but in the bent-tube case the phenomenon is of a purely quantum origin because there are no classical closed trajectories, apart from those given by a zero measure set of initial conditions in the phase space.

The main goal of the present work is a thorough analysis of the *essential spectrum* of (1.2). In particular, we find sufficient conditions which guarantee that the essential spectrum of a curved tube 'does not differ too much' from the straight case (for simplicity, we present here our results only for $d = 2$, see theorem 3.5 for the d-dimensional case):

Theorem 1.1 ($d = 2$)**.** *Let Γ be as above for $d = 2$ ($\kappa := \kappa_1$) and $\mathcal{T} := \{n^2\nu_1\}_{n=1}^\infty$ with $\nu_1 := \pi^2/(2a)^2$ (the set of eigenvalues of the Dirichlet Laplacian in the one-dimensional cross-section ω). Suppose*

1. $\kappa(s), \ddot{\kappa}(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\exists \vartheta \in (0, 1]$ *s.t.* $\dot{\kappa}(s)$, $\dddot{\kappa}(s) = \mathcal{O}(|s|^{-(1+\vartheta)})$.

Then

(i) $\sigma_{\mathrm{ess}}(-\Delta_{\mathrm{D}}^\Gamma) = [\nu_1, \infty)$,

(ii) $\sigma_{\mathrm{sc}}\big(-\Delta_{\mathrm{D}}^{\Gamma}\big) = \varnothing$,
(iii) $\sigma_{\mathrm{p}}\big(-\Delta_{\mathrm{D}}^{\Gamma}\big) \cup \mathcal{T}$ *is closed and countable,*
(iv) $\sigma_{\mathrm{p}}\big(-\Delta_{\mathrm{D}}^{\Gamma}\big) \setminus \mathcal{T}$ *is composed of finitely degenerated eigenvalues which can accumulate at points of* $\mathcal{T}$ *only.*

To prove this theorem (and the general theorem 3.5), we use the conjugate operator method introduced by Mourre [27] and lastly developed by Amrein *et al* [2]. Note that the set $\mathcal{T}$ plays a role analogous to the set of *thresholds* in the Mourre theory of N-body Schrödinger operators [9].

Actually, the property (i) holds true whenever the first curvature vanishes at infinity, without assuming any decay of the derivatives (they may not even exist), see [24] for $d = 2$ and [8] for the general case. Our second result (ii) can be compared only with [13] (see also [12]), where the problem of resonances is investigated for $d = 2$. Assuming that there exists $\vartheta \in (0, 1]$ such that $\kappa(s), \dot{\kappa}(s)^2, \ddot{\kappa}(s) = \mathcal{O}(|s|^{-(1+\vartheta)})$, the authors proved the absence of singular continuous spectrum as a consequence of the completeness of wave operators obtained by standard smooth perturbation methods of scattering theory. Note that our and their results are independent. Indeed, while we need to require a faster decay of $\dot{\kappa}$ and also impose a condition on $\dddot{\kappa}$, our decay assumptions on κ and $\ddot{\kappa}$ are in contrast much weaker. Our other spectral results (iii) and (iv) (and (ii) for $d \geqslant 3$) are new.

The organization of this paper is as follows. In section 2, we consider the Schrödinger-type operator

$$H := -\partial_i G^{ij} \partial_j + V \quad \text{on} \quad \mathcal{H}(\Omega) := L^2(\Omega) \tag{1.3}$$

subject to Dirichlet boundary conditions, i and j being summation indices taking values in $\{1, \ldots, d\}$, $G \equiv (G^{ij})$ a real symmetric matrix-valued measurable function on Ω and V the multiplication operator by a real-valued measurable function on Ω. We make assumptions 2.1 and 2.2 stated below. Adapting the approach of [3, 10] to non-zero V and G different from a multiple of the identity, we study the nature of the essential spectrum of the operator H. In particular, we prove the absence of singular continuous spectrum and state properties of possible embedded eigenvalues. The result is contained in theorem 2.16 and is of independent interest. In section 3, we apply it to the case of curved tubes (1.1). Using the diffeomorphism $\mathscr{L} : \Omega \to \Gamma$ and a unitary transformation (ideas which go back to [15]), we cast the Laplacian (1.2) into a unitarily equivalent operator of the form (1.3) for which theorem 2.16 can be used. The obtained spectral results can be found in theorem 3.5 (the general version of theorem 1.1). Finally, in section 4, we similarly investigate the essential spectrum of the Dirichlet Laplacian in an infinite strip in an abstract two-dimensional Riemannian manifold of curvature K. The general result is contained in theorem 4.2, while the case of flat strips, i.e. with $K = 0$, is summarized in theorem 4.3 (the latter involves the curved strips in $\mathbb{R}^2$ as a special case).

For the conjugate operator method and notation used in section 2, the reader is referred to [2] and particularly to short well-arranged reviews of the abstract theory in [3, section 2] or [10, section 1]. A more detailed geometric background for sections 3 and 4 can be found in [8, 22] and [18, 23], respectively.

We use the standard component notation of tensor analysis throughout the paper. In particular, the repeated indices convention is adopted henceforth, the range of indices being $1, \ldots, d$ for Latin and $2, \ldots, d$ for Greek. The indices are associated in a natural way with the components of $x \in \mathbb{R} \times \omega$. The partial derivative w.r.t. x^i is often denoted by a comma with the index i. The brackets $(\cdot)$ are used in order to distinguish a matrix from its coefficients. The symbols δ_{ij} and δ^{ij} are reserved for the components of the identity matrix 1.

2. Schrödinger-type operators in straight tubes

2.1. Preliminaries

Let ω be an (arbitrary) bounded open connected set in $\mathbb{R}^{d-1}$, $d \geqslant 2$, and consider the straight tube $\Omega := \mathbb{R} \times \omega$. Our object of interest in this section is the operator given formally by (1.3), subject to Dirichlet boundary conditions. In addition to the basic properties required for the matrix G and function V, we make the following assumptions.

Assumption 2.1.

1. $\exists C_\pm \in (0, \infty)$ *s.t.* $C_- 1 \leqslant G(x) \leqslant C_+ 1$ *for a.e.* $x \in \Omega$,
2. $\forall i, j \in \{1, \ldots, d\}$, $\lim_{R\to\infty} \operatorname{ess\,sup}_{x\in(\mathbb{R}\setminus[-R,R])\times\omega} \left|G^{ij}(x) - \delta^{ij}\right| = 0$,
3. $\exists \vartheta_1 \in (0, 1]$, $C \in (0, \infty)$ *s.t.* $\left(\left|G^{ij}{}_{,1}(x)\right|\right) \leqslant C \langle x^1\rangle^{-(1+\vartheta_1)} 1$ *for a.e.* $x \in \Omega$,
4. $G^{1i}{}_{,i} \in L^\infty(\Omega)$.

Here $\langle\cdot\rangle := (1 + |\cdot|^2)^{1/2}$ and the inequalities must be understood in the sense of matrices.

Assumption 2.2.

1. $V \in L^\infty(\Omega)$,
2. $\lim_{R\to\infty} \operatorname{ess\,sup}_{x\in(\mathbb{R}\setminus[-R,R])\times\omega} |V(x)| = 0$,
3. $\exists \vartheta_2 \in (0, 1]$, $C \in (0, \infty)$ *s.t.* $|V_{,1}(x)| \leqslant C \langle x^1\rangle^{-(1+\vartheta_2)}$ *for a.e.* $x \in \Omega$.

Let us fix some notations. We write $\mathcal{H}^\nu(\Omega)$ and $\mathcal{H}^\nu_0(\Omega)$, $\nu \in \mathbb{R}$, for the usual Sobolev spaces [1]. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathscr{B}(\mathcal{H}_1, \mathcal{H}_2)$, respectively $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, the set of bounded, respectively compact, operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We also define $\mathscr{B}(\mathcal{H}_1) := \mathscr{B}(\mathcal{H}_1, \mathcal{H}_1)$ and $\mathcal{K}(\mathcal{H}_1) := \mathcal{K}(\mathcal{H}_1, \mathcal{H}_1)$. We denote by $\mathcal{H}_1^*$ the topological antidual of $\mathcal{H}_1$. We write $(\cdot, \cdot)$ for the inner product in $\mathcal{H}(\Omega)$ and $\|\cdot\|$ for the norm in $\mathcal{H}(\Omega)$ and $\mathscr{B}(\mathcal{H}(\Omega))$.

We now give a meaning to the formal expression (1.3). We start by introducing the sesquilinear form Q_0 on $\mathcal{H}(\Omega)$ defined by

$$Q_0(\varphi, \psi) := (\varphi_{,i}, \delta^{ij}\psi_{,j}) \qquad \varphi, \psi \in \mathcal{D}(Q_0) := \mathcal{H}^1_0(\Omega) \tag{2.1}$$

which is densely defined, symmetric, non-negative and closed. Consequently, there exists a unique self-adjoint operator H_0 associated with it, which is just the Dirichlet Laplacian $-\Delta^\Omega_{\mathrm{D}}$ on $L^2(\Omega)$. We have

$$H_0\psi = -\Delta\psi \qquad \psi \in \mathcal{D}(H_0) = \left\{\psi \in \mathcal{H}^1_0(\Omega) : \Delta\psi \in \mathcal{H}(\Omega)\right\}.$$

We consider H as an operator obtained by perturbing the free Hamiltonian H_0. Since the matrix G is uniformly positive and bounded by assumption 2.1.1, the sesquilinear form $(\varphi, \psi) \mapsto (\varphi_{,i}, G^{ij}\psi_{,j})$, defined on $\mathcal{D}(Q_0) \times \mathcal{D}(Q_0)$, is also densely defined, symmetric, non-negative and closed. At the same time, the potential V is supposed to be bounded by assumption 2.2.1, which means that the sesquilinear form Q defined by

$$Q(\varphi, \psi) := (\varphi_{,i}, G^{ij}\psi_{,j}) + (\varphi, V\psi) \qquad \varphi, \psi \in \mathcal{D}(Q) := \mathcal{H}^1_0(\Omega) \tag{2.2}$$

gives rise to a semi-bounded self-adjoint operator H. Using the representation theorem [21, chapter VI, theorem 2.1] and the fact that V is bounded (recall also assumption 2.1.1), one may check that

$$\mathcal{D}(H) = \left\{\psi \in \mathcal{H}^1_0(\Omega) : \partial_i G^{ij}\partial_j\psi \in \mathcal{H}(\Omega)\right\}$$

where the derivatives must be interpreted in the distributional sense, and that H is acting as in (1.3) on its domain.

For any $z \in \mathbb{C}\setminus\sigma(H_0)$, respectively $z \in \mathbb{C}\setminus\sigma(H)$, let $R_0(z) := (H_0 - z)^{-1}$, respectively $R(z) := (H - z)^{-1}$.

2.2. *Localization of the essential spectrum*

The Dirichlet Laplacian $-\Delta_D^\omega$ on $L^2(\omega)$, i.e. the operator associated with

$$q(\varphi,\psi) := (\varphi_{,\mu},\delta^{\mu\nu}\psi_{,\nu}) \qquad \varphi,\psi\in\mathcal{D}(q) := \mathcal{H}_0^1(\omega)$$

has a purely discrete spectrum consisting of eigenvalues $\nu_1 < \nu_2 \leqslant \nu_3 \leqslant \cdots$ with $\nu_1 > 0$. We set $\mathcal{T} := \{\nu_n\}_{n=1}^\infty$. Since H_0 is naturally decoupled in the following way:

$$H_0 = -\Delta^{\mathbb{R}} \otimes 1 + 1 \otimes \left(-\Delta_D^\omega\right) \quad \text{on} \quad L^2(\mathbb{R}) \otimes L^2(\omega)$$

where '$\otimes$' denotes the closed tensor product, 1 denotes the identity operators on appropriate spaces and $-\Delta^{\mathbb{R}}$ is the Laplacian on $L^2(\mathbb{R})$, one has

$$\sigma(H_0) = \sigma_{\text{ess}}(H_0) = [\nu_1,\infty). \tag{2.3}$$

In order to prove that (under our assumptions) H possesses the same essential spectrum, we need the following lemma.

Lemma 2.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$ and set $\phi := \varphi\otimes 1$ on Ω. Then, as a multiplication operator, $\phi \in \mathcal{K}\big(\mathcal{D}(H_0), \mathcal{H}_0^1(\Omega)\big)$.*

Proof. Since

$$\phi = H_0^{-1/2} H_0^{1/2} \phi H_0^{-1} H_0$$

in $\mathscr{B}(\mathcal{D}(H_0), \mathcal{H}_0^1(\Omega))$, $H_0 \in \mathscr{B}(\mathcal{D}(H_0), \mathcal{H}(\Omega))$ and $H_0^{-1/2} \in \mathscr{B}\left(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\right)$, it is enough to prove that $H_0^{1/2}\phi H_0^{-1} \in \mathcal{K}(\mathcal{H}(\Omega))$. However,

$$\begin{aligned} H_0^{1/2}\phi H_0^{-1} &= H_0^{-1/2}[H_0,\phi]H_0^{-1} + H_0^{-1/2}\phi \\ &= -H_0^{-1/2}(2\phi_{,1}\partial_1 + \phi_{,11})H_0^{-1} + H_0^{-1/2}\phi \end{aligned} \tag{2.4}$$

where each term on the rhs is in $\mathcal{K}(\mathcal{H}(\Omega))$. Let us demonstrate it for the first term. Since $\partial_1 H_0^{-1} \in \mathscr{B}(\mathcal{H}(\Omega))$, it is sufficient to prove that $H_0^{-1/2}\phi_{,1} \in \mathcal{K}(\mathcal{H}(\Omega))$. Let $z_1 \in (-\infty, 0)$ and $z_2 \in (-\infty,\nu_1)$ be such that $z_1 + z_2 = 0$. Define $R_{\shortparallel}(z_1) := (-\Delta^{\mathbb{R}} - z_1)^{-1}$ and $R_\perp(z_2) := \left(-\Delta_D^\omega - z_2\right)^{-1}$. Then, using some standard results on tensor products of operators [20, chapter 11], one can write

$$H_0^{-1/2}\phi_{,1} = H_0^{-1/2}\big[R_{\shortparallel}^{-1/4}(z_1)\otimes R_\perp^{-1/4}(z_2)\big]\big[R_{\shortparallel}^{1/4}(z_1)\varphi_{,1}\otimes R_\perp^{1/4}(z_2)\big]$$

where $\varphi_{,1}$ is viewed as a multiplication operator in $L^2(\mathbb{R})$. The third factor on the rhs is in $\mathcal{K}(\mathcal{H}(\Omega))$ because $-\Delta_D^\omega$ has a compact resolvent and $R_{\shortparallel}^{1/4}(z_1)\varphi_{,1} \in \mathcal{K}(L^2(\mathbb{R}))$ by [2, theorem 4.1.3]. The remaining factors can be rewritten as

$$\Psi(X_1,X_2) := (X_1+X_2)^{-1/2} X_1^{1/4} X_2^{1/4}$$

with $X_1 := (-\Delta^{\mathbb{R}} - z_1)\otimes 1$ and $X_2 := 1\otimes\left(-\Delta_D^\omega - z_2\right)$ (both self-adjoint and mutually commuting). So, one can estimate

$$\|\Psi(X_1,X_2)\| \leqslant \sup_{x_1,x_2\in(0,\infty)} (x_1+x_2)^{-1/2}(x_1x_2)^{1/4} < \infty.$$

Hence, the first term on the rhs of (2.4) is in $\mathcal{K}(\mathcal{H}(\Omega))$. The argument is similar for the remaining terms. □

Proposition 2.2. *One has*

(i) $\forall z \in \mathbb{C}\backslash(\sigma(H)\cup\sigma(H_0))$, $R(z) - R_0(z) \in \mathcal{K}(\mathcal{H}(\Omega))$,
(ii) $\sigma_{\text{ess}}(H) = [\nu_1,\infty)$.

Proof. We prove (i) for some (and hence for all) values of $z \in \mathbb{C}\backslash(\sigma(H) \cup \sigma(H_0))$. Let $z \in \mathbb{C}\backslash\mathbb{R}$. Define $R_1(z) := (H_0 + V - z)^{-1}$. Then, one has

$$R(z) - R_0(z) = R(z) - R_1(z) - R_1(z) V R_0(z).$$

Let us first consider $R(z) - R_1(z)$. Knowing that H and $H_0 + V$ have the same form domain, the identity

$$R(z) - R_1(z) = -R(z)(H - H_0 - V) R_1(z)$$

holds in $\mathscr{B}\left(\mathcal{H}^{-1}(\Omega), \mathcal{H}_0^1(\Omega)\right)$. But, one has the following sequence of continuous and dense embeddings of Hilbert spaces

$$\mathcal{D}(H) \subset \mathcal{H}_0^1(\Omega) \subset \mathcal{H}(\Omega) \subset \mathcal{H}^{-1}(\Omega) \subset \mathcal{D}(H)^*$$

which implies that $R(z)$ extends (by duality) to a homeomorphism of $\mathcal{D}(H)^*$ onto $\mathcal{H}(\Omega)$. Thus, since $R_1(z)$ is also a homeomorphism from $\mathcal{H}(\Omega)$ onto $\mathcal{D}(H_0)$, $R(z) - R_1(z) \in \mathcal{K}(\mathcal{H}(\Omega))$ if and only if $H - H_0 - V \in \mathcal{K}(\mathcal{D}(H_0), \mathcal{D}(H)^*)$. For all $n \in \mathbb{N}\backslash\{0\}$, let $\varphi_n \in C_0^\infty(\mathbb{R})$ be such that $0 \leqslant \varphi_n \leqslant 1$ and

$$\varphi_n(x^1) = \begin{cases} 1 & \text{if} \quad |x^1| \leqslant n \\ 0 & \text{if} \quad |x^1| \geqslant n + 1. \end{cases}$$

Set $\phi_n := \varphi_n \otimes 1$ on Ω and

$$K_n \psi := -\partial_i F^{ij} \phi_n \partial_j \psi \qquad \psi \in \mathcal{D}(H_0)$$

where $(F^{ij}) := (G^{ij} - \delta^{ij})$. Clearly, $H - H_0 - V, K_n \in \mathscr{B}(\mathcal{D}(H_0), \mathcal{D}(H)^*)$ and

$$\begin{aligned} &\|K_n - (H - H_0 - V)\|_{\mathscr{B}(\mathcal{D}(H_0),\mathcal{D}(H)^*)} \\ &\qquad \equiv \sup_{\psi \in \mathcal{D}(H_0), \|\psi\|_{\mathcal{D}(H_0)}=1} \|(1 + H^2)^{-1/2}[-\partial_i F^{ij} (\phi_n - 1)\partial_j]\psi\| \\ &\qquad \leqslant \sup_{\psi \in \mathcal{D}(H_0), \|\psi\|_{\mathcal{D}(H_0)}=1} \sum_{j=1}^{d} \|(1 + H^2)^{-1/2}\partial_i\| \|F^{ij} (\phi_n - 1)\|_\infty \|\psi\|_{\mathcal{H}_0^1(\Omega)} \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

where we have used the fact that $\mathcal{D}(H_0) \subset \mathcal{H}_0^1(\Omega)$ continuously and assumption 2.1.2 in the final step. So, it only remains to show that $K_n \in \mathcal{K}(\mathcal{D}(H_0), \mathcal{D}(H)^*)$. After a commutation, one gets in $\mathscr{B}(\mathcal{D}(H_0), \mathcal{D}(H)^*)$

$$K_n = -\partial_i F^{ij} \partial_j \phi_n + \partial_i F^{i1} \phi_{n,1}$$

where $\phi_n, \phi_{n,1}$ are seen as multiplication operators in $\mathcal{H}(\Omega)$. It is clear that both $\partial_i F^{ij} \partial_j$ and $\partial_i F^{i1}$ are in $\mathscr{B}\left(\mathcal{H}_0^1(\Omega), \mathcal{D}(H)^*\right)$. Moreover, ϕ_n and $\phi_{n,1}$ are in $\mathcal{K}\left(\mathcal{D}(H_0), \mathcal{H}_0^1(\Omega)\right)$ by lemma 2.1. Thus, $K_n \in \mathcal{K}(\mathcal{D}(H_0), \mathcal{D}(H)^*)$ so that $R(z) - R_1(z) \in \mathcal{K}(\mathcal{H}(\Omega))$. Using similar arguments, one can also prove that $R_1(z)V\phi_n R_0(z)$ is compact and converges to $R_1(z)V R_0(z)$ in $\mathscr{B}(\mathcal{H}(\Omega))$ due to assumption 2.2.2. This implies that $R_1(z) V R_0(z) \in \mathcal{K}(\mathcal{H}(\Omega))$.

(ii) It is a direct consequence of (i), (2.3) and Weyl's theorem [29, theorem XIII.14]. □

Remark 2.3. Note that assumptions 2.1.3, 2.1.4 and 2.2.3 are not used in the proof of proposition 2.2.

2.3. Nature of the essential spectrum

This part is devoted to a more detailed analysis of the essential spectrum of H. In particular, we show that the singular continuous spectrum is empty. The strategy adapted from [3] is the following. Firstly, we construct a dilation operator A such that $H_0 \in C^\infty(A)$ and $H \in C^{1+\vartheta}(A)$ with $\vartheta := \min\{\vartheta_1, \vartheta_2\} \in (0, 1]$ (see [2], [3, section 2] or [10, section 1] for definitions of the spaces involved here and in the following). Secondly, we prove that A is *strictly conjugate* (in Mourre's sense) to H_0 on $\mathbb{R}\backslash\mathcal{T}$. Finally, since $R(i) - R_0(i)$ is compact by the first claim of proposition 2.2 and both H and H_0 are of class $C_u^1(A) \supseteq C^{1+\vartheta}(A) \supseteq C^\infty(A)$, it follows that A is conjugate to H on $\mathbb{R}\backslash\mathcal{T}$ as well.

2.3.1. The dilation operator. Let q^1 be the multiplication operator by the coordinate x^1 in $\mathcal{H}(\Omega)$. Let

$$A := \tfrac{1}{2}(q^1 p_1 + p_1 q^1) \qquad \text{with} \quad p_1 := -i\,\partial_1 \tag{2.5}$$

be the dilation operator in $\mathcal{H}(\Omega)$ w.r.t. x^1, i.e. the self-adjoint extension of the operator defined by expression (2.5) with $C_0^\infty(\Omega)$ as initial domain. Define $A_{\shortparallel}$ as the self-adjoint operator in $L^2(\mathbb{R})$ such that $A = A_{\shortparallel} \otimes 1$.

Remark 2.4. The group $\{\mathrm{e}^{\mathrm{i}At}\}_{t\in\mathbb{R}}$ leaves invariant $\mathcal{H}_0^1(\Omega)$. Indeed, using the natural isomorphism $\mathcal{H}_0^1(\Omega) \simeq \mathcal{H}^1(\mathbb{R}) \otimes \mathcal{H}_0^1(\omega)$, one can write

$$\forall t \in \mathbb{R}, \qquad \mathrm{e}^{\mathrm{i}At}\mathcal{H}_0^1(\Omega) = (\mathrm{e}^{\mathrm{i}A_{\shortparallel}t}\mathcal{H}^1(\mathbb{R})) \otimes \mathcal{H}_0^1(\omega).$$

Then, the affirmation follows from the fact [2, proposition 4.2.4] that $\mathcal{H}^1(\mathbb{R})$ is stable under $\{\mathrm{e}^{\mathrm{i}A_{\shortparallel}t}\}_{t\in\mathbb{R}}$.

In order to deal with the commutator $\mathrm{i}[H, A]$, we need the following family of operators,

$$\{p_1(\varepsilon) := p_1(1 + \mathrm{i}\varepsilon p_1)^{-1}\}_{\varepsilon>0} \tag{2.6}$$

which regularizes the momentum operator p_1:

Lemma 2.5. *One has*

(i) $\{p_1(\varepsilon)\}_{\varepsilon>0} \subset \mathscr{B}(\mathcal{H}(\Omega))$,
(ii) $\{p_1(\varepsilon)\}_{\varepsilon>0}$ *is uniformly bounded in* $\mathscr{B}(\mathcal{H}^1(\Omega), \mathcal{H}(\Omega))$ *and* $s\text{-}\lim_{\varepsilon\to 0} p_1(\varepsilon) = p_1$ *in* $\mathscr{B}(\mathcal{H}^1(\Omega), \mathcal{H}(\Omega))$,
(iii) $\forall \varepsilon > 0$, $[p_1(\varepsilon), q_1] = -\mathrm{i}(1 + \mathrm{i}\varepsilon p_1)^{-2}$ *in* $\mathscr{B}(\mathcal{H}(\Omega))$,
(iv) $\forall \varepsilon > 0$, $p_1(\varepsilon)\mathcal{H}_0^1(\Omega) \subset \mathcal{H}_0^1(\Omega)$.

Proof. The first three assertions are established in [3, lemma 4.1]. Consequently, it only remains to prove the last statement. Using the isomorphism mentioned in remark 2.4, one can write

$$\forall \varepsilon > 0, \qquad p_1(\varepsilon)\mathcal{H}_0^1(\Omega) = -\mathrm{i}\varepsilon^{-1}\{[1 + \mathrm{i}\varepsilon^{-1}(p_1 - \mathrm{i}\varepsilon^{-1})^{-1}]\mathcal{H}^1(\mathbb{R})\} \otimes \mathcal{H}_0^1(\omega)$$

where p_1 on the rhs must be viewed as an operator acting in $L^2(\mathbb{R})$. With this last relation, it is clear that $\mathcal{H}_0^1(\Omega)$ is left invariant by the family $\{p_1(\varepsilon)\}_{\varepsilon>0}$. □

We also need the following density result for the set $\mathcal{D}(H)_c := \{\psi \in \mathcal{D}(H) : \mathrm{supp}(\psi)$ is compact$\}$.

Lemma 2.6. *One has*

(i) $\mathcal{D}(H)_c$ *is dense in* $\mathcal{D}(H)$
(ii) $\mathcal{D}(H)_c$ *is dense in* $\mathcal{H}_0^1(\Omega)$.

Proof. (i) We are inspired by [10, lemma 2.1]. Let $\psi \in \mathcal{D}(H)$. Define $\varphi_0 \in C_0^\infty(\mathbb{R})$ such that

$$\varphi_0(x^1) = \begin{cases} 1 & \text{if} \quad |x^1| \leqslant 1 \\ 0 & \text{if} \quad |x^1| \geqslant 2. \end{cases}$$

Let $n \in \mathbb{N}$. Set $\varphi_n(x^1) := \varphi_0(x^1/(n+1))$ for $x^1 \in \mathbb{R}$ and $\phi_n := \varphi_n \otimes 1$ on Ω. Then, $\phi_n\psi \in \mathcal{H}_0^1(\Omega)$, $\lim_{n\to\infty} \phi_n\psi = \psi$ in $\mathcal{H}(\Omega)$ and

$$H\phi_n\psi = \phi_n H\psi - 2\phi_{n,1} G^{1j}\psi_{,j} - \phi_{n,11}G^{11}\psi - \phi_{,1}G^{1i}{}_{,i}\psi \tag{2.7}$$

in the sense of distributions. Using the fact that $\mathrm{supp}(\phi_n)$ is compact, assumptions 2.1.1 and 2.1.4, one has $\phi_n\psi \in \mathcal{D}(H)_\mathrm{c}$. Moreover, as a consequence of (2.7) and the property

$$\forall k \in \mathbb{N}, \forall x \in \Omega \qquad \partial_1^k \phi_n(x) = (n+1)^{-k}\varphi_0^{(k)}(x^1/(n+1))$$

one also has $\lim_{n\to\infty} H\phi_n\psi = H\psi$ in $\mathcal{H}(\Omega)$.

(ii) Using point (i) and the fact that $\mathcal{D}(H) \subset \mathcal{H}_0^1(\Omega)$ continuously and densely, one gets the following embeddings,

$$\mathcal{H}_0^1(\Omega) = \overline{\overline{\mathcal{D}(H)_\mathrm{c}}^{\mathcal{D}(H)}}^{\mathcal{H}_0^1(\Omega)} \subseteq \overline{\overline{\mathcal{D}(H)_\mathrm{c}}^{\mathcal{H}_0^1(\Omega)}}^{\mathcal{H}_0^1(\Omega)} = \overline{\mathcal{D}(H)_\mathrm{c}}^{\mathcal{H}_0^1(\Omega)} \subseteq \mathcal{H}_0^1(\Omega)$$

which, in particular, imply that $\mathcal{D}(H)_\mathrm{c}$ is dense in $\mathcal{H}_0^1(\Omega)$. □

Now, we can compute the commutator $\mathrm{i}[H, A]$.

Proposition 2.7. *The sesquilinear form $\mathcal{Q}$ on $\mathcal{H}(\Omega)$ defined by*

$$\mathcal{Q}(\varphi, \psi) := \mathrm{i}[(H\varphi, A\psi) - (A\varphi, H\psi)] \qquad \varphi, \psi \in \mathcal{D}(\mathcal{Q}) := \mathcal{D}(H) \cap \mathcal{D}(A)$$

is continuous on $\mathcal{D}(H)_\mathrm{c}$ for the topology induced by $\mathcal{H}_0^1(\Omega)$. Moreover,

$$\mathrm{i}[H, A] = -\partial_j G^{1j}\partial_1 - \partial_1 G^{1j}\partial_j + \partial_i q^1 G^{ij}{}_{,1}\partial_j - q^1 V_{,1} \tag{2.8}$$

as operators in $\mathscr{B}\left(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\right)$.

Proof. Let $\varphi, \psi \in \mathcal{D}(H)_c$. Using the identity $A = q^1 p_1 - \frac{\mathrm{i}}{2}$ valid on $\mathcal{D}(H)_c \subset \mathcal{D}(A)$, we have

$$\begin{aligned} \mathcal{Q}(\varphi, \psi) &= \mathrm{i}[(H\varphi, A\psi) - (A\varphi, H\psi)] \\ &= (\varphi, H\psi) + \mathrm{i}[(-\partial_i G^{ij}\partial_j\varphi, q^1 p_1\psi) - (q^1 p_1\varphi, -\partial_i G^{ij}\partial_j\psi)] \\ &\quad + (V\varphi, q^1\psi_{,1}) + (q^1\varphi_{,1}, V\psi). \end{aligned}$$

In order to justify the subsequent integration by parts, we employ the family (2.6). Since ψ has a compact support and belongs to $\mathcal{H}_0^1(\Omega)$, it follows by using properties (iii) and (iv) of lemma 2.5 that $q^1 p_1(\varepsilon)\psi \in \mathcal{H}_0^1(\Omega)$ for all $\varepsilon > 0$. So, we can write

$$\begin{aligned} (-\partial_i G^{ij}\partial_j\varphi, q^1 p_1\psi) &= \lim_{\varepsilon\to 0}(-\partial_i G^{ij}\partial_j\varphi, q^1 p_1(\varepsilon)\psi) \\ &= \lim_{\varepsilon\to 0}(\varphi_{,j}, G^{ij}\partial_i q^1 p_1(\varepsilon)\psi) \\ &= -\mathrm{i}(\varphi_{,j}, G^{1j}\psi_{,1}) + \lim_{\varepsilon\to 0}(\varphi_{,i}, G^{ij}q^1 p_1(\varepsilon)\psi_{,j}) \end{aligned}$$

and similarly for the integral

$$(q^1 p_1\varphi, -\partial_i G^{ij}\partial_j\psi) = \mathrm{i}(\varphi_{,1}, G^{1j}\psi_{,j}) + \lim_{\varepsilon\to 0}(p_1(\varepsilon)^*\varphi_{,i}, q^1 G^{ij}\psi_{,j}).$$

Since

$$\lim_{\varepsilon\to 0}(p_1(\varepsilon)^*\varphi_{,i}, q^1G^{ij}\psi_{,j}) = \lim_{\varepsilon\to 0}(\varphi_{,i}, p_1(\varepsilon)q^1G^{ij}\psi_{,j})$$
$$= -\mathrm{i}\big[(\varphi_{,i}, G^{ij}\psi_{,j}) + \big(\varphi_{,i}, q^1G^{ij}{}_{,1}\psi_{,j}\big)\big] + \lim_{\varepsilon\to 0}(\varphi_{,i}, q^1G^{ij}p_1(\varepsilon)\psi_{,j})$$

and

$$(q^1\varphi_{,1}, V\psi) = -(\varphi, \partial_1 q^1 V\psi) = -(\varphi, V\psi) - (\varphi, q^1V_{,1}\psi) - (\varphi, q^1V\psi_{,1})$$

we finally obtain that

$$\mathcal{Q}(\varphi,\psi) = (\varphi_{,j}, G^{1j}\psi_{,1}) + (\varphi_{,1}, G^{1j}\psi_{,j}) - \big(\varphi_{,i}, q^1G^{ij}{}_{,1}\psi_{,j}\big) - (\varphi, q^1V_{,1}\psi). \tag{2.9}$$

This implies that $\mathcal{Q}$ restricted to $\mathcal{D}(H)_c$ is continuous for the topology induced by $\mathcal{H}_0^1(\Omega)$. Now, $\mathcal{D}(H)_c$ is dense in $\mathcal{H}_0^1(\Omega)$ by lemma 2.6(ii). Thus, $\mathcal{Q}$ defines (by continuous extension) an operator in $\mathscr{B}\big(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\big)$, which we shall denote $\mathrm{i}[H, A]$. Furthermore, using (2.9), we obtain (2.8) in $\mathscr{B}\big(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\big)$. □

2.3.2. Strict Mourre estimate for the free Hamiltonian. Now we prove that H_0 is of class $C^\infty(A)$ and A is strictly conjugate to it on $\mathbb{R}\backslash\mathcal{T}$. So, let us first recall the following definition [2, section 7.2.1 & 7.2.2]:

Definition 2.8. *Let A, H be self-adjoint operators in a Hilbert space $\mathcal{H}$ with H of class $C^1(A)$. Furthermore, if $S, T \in \mathscr{B}(\mathcal{H})$, we write $S \gtrsim T$ if there exists $K \in \mathcal{K}(\mathcal{H})$ so that $S \geqslant T + K$. Then, $\forall\lambda \in \mathbb{R}$,*

$$\varrho_H^A(\lambda) := \sup\{a \in \mathbb{R} : \exists\varepsilon > 0\ s.t.\ E^H(\lambda;\varepsilon)\mathrm{i}[H, A]E^H(\lambda;\varepsilon) \geqslant a E^H(\lambda;\varepsilon)\}$$
$$\widetilde{\varrho}_H^A(\lambda) := \sup\{a \in \mathbb{R} : \exists\varepsilon > 0\ s.t.\ E^H(\lambda;\varepsilon)\mathrm{i}[H, A]E^H(\lambda;\varepsilon) \gtrsim a E^H(\lambda;\varepsilon)\}$$

where $E^H(\lambda;\varepsilon) := E^H((\lambda-\varepsilon, \lambda+\varepsilon))$ designates the spectral projection of H for the interval $(\lambda-\varepsilon, \lambda+\varepsilon)$.

We also need the following natural generalization of [5, theorem 3.4].

Theorem 2.9. *Let H_1, H_2 be two self-adjoint, bounded from below operators in the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Assume that A_j, $j = 1, 2$, is a self-adjoint operator in $\mathcal{H}_j$ such that H_j is of class $C^k(A_j)$, $k \in (\mathbb{N}\backslash\{0\}) \cup \{+\infty\}$. Let $H := H_1 \otimes 1 + 1 \otimes H_2$ and $A := A_1 \otimes 1 + 1 \otimes A_2$, which are self-adjoint operators in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then H is of class $C^k(A)$ and $\forall\lambda \in \mathbb{R}$,*

$$\varrho_H^A(\lambda) = \inf_{\lambda=\lambda_1+\lambda_2}\big[\varrho_{H_1}^{A_1}(\lambda_1) + \varrho_{H_2}^{A_2}(\lambda_2)\big].$$

Corollary 2.10. *$H_0 \in C^\infty(A)$ and*

$$\forall\lambda \in \mathbb{R}, \qquad \varrho_{H_0}^A(\lambda) = \begin{cases} 2\rho(\lambda) & \text{if } \lambda \geqslant \nu_1 \\ +\infty & \text{if } \lambda < \nu_1 \end{cases} \tag{2.10}$$

where $\rho(\lambda) := \lambda - \sup\{\zeta \in \mathcal{T} : \zeta \leqslant \lambda\}$ is strictly positive on $\mathbb{R}\backslash\mathcal{T}$.

Proof. $A_1 := A_{\shortparallel}$, $A_2 := 0$ are self-adjoint in $L^2(\mathbb{R})$, respectively $L^2(\omega)$. $H_1 := p_1^2$, $H_2 := -\Delta_D^\omega$ are self-adjoint, bounded from below in $L^2(\mathbb{R})$, respectively $L^2(\omega)$. Clearly, $p_1^2 \in C^\infty(A_{\shortparallel})$ and $-\Delta_D^\omega \in C^\infty(0)$ [2, example 6.2.8]. The first part of the claim and (2.10) then follows from theorem 2.9. The expression for $\rho(\lambda)$ is a direct consequence of the respective behaviours of [2, section 7.2.1] $\varrho_{p_1^2}^{A_{\shortparallel}}$ and $\varrho_{-\Delta_D^\omega}^0$:

$$\begin{bmatrix} \varrho_{p_1^2}^{A_{\shortparallel}}(\lambda_1) \\ \varrho_{-\Delta_D^\omega}^0(\lambda_2) \end{bmatrix} = \begin{cases} \begin{bmatrix} 2\lambda_1 \\ +\infty \end{bmatrix} & \text{if } \begin{bmatrix} \lambda_1 \geqslant 0 \\ \lambda_1 < 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ +\infty \end{bmatrix} & \text{if } \begin{bmatrix} \lambda_2 \in \mathcal{T} \\ \lambda_2 \in \mathbb{R}\backslash\mathcal{T} \end{bmatrix}. \end{cases}$$

□

2.3.3. Regularity of the Hamiltonian. In order to prove the regularity of H, we need two technical lemmas.

Lemma 2.11. $\forall z \in \mathbb{R}\backslash\sigma(H)$, $\forall\vartheta \leqslant 1$, *one has*

(i) $[R(z), \langle q^1\rangle^\vartheta] \in \mathscr{B}\big(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\big)$,

(ii) $\forall i \in \{1, \ldots, d\}$, $[R(z), \langle q^1\rangle^\vartheta]\partial_i \in \mathscr{B}\big(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\big)$.

This is established by adapting the proof of [3, lemma 4.3] while the next lemma follows from the use of [3, proof of proposition 4.2].

Lemma 2.12. *Let* $S \in \mathscr{B}(\mathcal{H}(\Omega))$ *be self-adjoint and* $\vartheta \in (0, 1]$, *then*

$$\langle q^1\rangle^\vartheta S \in \mathscr{B}(\mathcal{H}(\Omega), \mathcal{H}^\vartheta(\mathbb{R}) \otimes L^2(\omega)) \quad\Longrightarrow\quad S \in C^\vartheta(A).$$

(Note that the proof involves principally two facts. First, $S \in \mathscr{B}(\mathcal{H}(\Omega), \mathcal{D}(|A|^\vartheta))$ implies that $S \in C^\vartheta(A)$. Second, the continuous embedding $\mathcal{H}_\vartheta^\vartheta(\mathbb{R}) \subseteq \mathcal{D}(|A_{\shortparallel}|^\vartheta)$, which follows by real interpolation [2, section 2.7] from the continuous embedding $\mathcal{H}_1^1(\mathbb{R}) \subseteq \mathcal{D}(|A_{\shortparallel}|)$.)

Remark 2.13. The facts that $\mathrm{i}[H, A] \in \mathscr{B}\big(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\big)$ and that $\mathcal{H}_0^1(\Omega)$ is stable under $\{\mathrm{e}^{\mathrm{i}At}\}_{t\in\mathbb{R}}$ imply [2, section 6.3] that $H \in C^1(A)$.

Proposition 2.14. $\exists\vartheta \in (0, 1]$ *such that* $H \in C^{1+\vartheta}(A)$.

Proof. We show that each term appearing in the expression for $B := \mathrm{i}[H, A]$ is at least of class $C^\gamma(A)$ for a certain $\gamma \in (0, 1]$.

Consider first $B_1 := -\partial_j G^{j1}\partial_1 - \partial_1 G^{1j}\partial_j$. An explicit calculation (analogous to that of the proof of proposition 2.7) implies that

$$\mathrm{i}[B_1, A] = -2\partial_1 G^{11}\partial_1 - \partial_1 G^{1j}\partial_j - \partial_j G^{j1}\partial_1 + \partial_j q^1 G^{j1}{}_{,1}\partial_1 + \partial_1 q^1 G^{1j}{}_{,1}\partial_j$$

as operators in $\mathscr{B}\big(\mathcal{H}_0^1(\Omega), \mathcal{H}^{-1}(\Omega)\big)$. Thus, $B_1 \in C^1(A)$ by remark 2.13.

Let $z \in \mathbb{R}\backslash\sigma(H)$. As a consequence of the fact that $H \in C^1(A)$, one can interpret $\mathrm{i}[A, R(z)]$ as the product of [2, section 6.2.2] three bounded operators, viz $R(z) : \mathcal{H}(\Omega) \to \mathcal{D}(H)$, $B : \mathcal{D}(H) \to \mathcal{D}(H)^*$ and $R(z) : \mathcal{D}(H)^* \to \mathcal{H}(\Omega)$. Thus, using proposition 2.7, one can write as an operator identity in $\mathscr{B}(\mathcal{H}(\Omega))$

$$\mathrm{i}[A, R(z)] = R(z)BR(z) = R(z)B_1R(z) + R(z)\partial_i q^1 G^{ij}{}_{,1}\partial_j R(z) - R(z)q^1 V_{,1}R(z).$$

Since the first term has already been shown to be bounded, it is enough to prove that the second and third terms on the rhs are of class $C^\gamma(A)$ for some $\gamma \in (0, 1]$.

We employ lemma 2.12 with $\vartheta := \min\{\vartheta_1, \vartheta_2\}$ in order to deal with both terms. Using some commutation relations, we get

$$\begin{aligned}\langle q^1\rangle^\vartheta R(z)\partial_i q^1 G^{ij}{}_{,1}\partial_j R(z) &= R(z)\partial_i\langle q^1\rangle^\vartheta q^1 G^{ij}{}_{,1}\partial_j R(z)\\ &\quad - [R(z), \langle q^1\rangle^\vartheta]\partial_i q^1 G^{ij}{}_{,1}\partial_j R(z) - R(z)[\partial_i, \langle q^1\rangle^\vartheta]q^1 G^{ij}{}_{,1}\partial_j R(z).\end{aligned}$$

Under assumption 2.1.3, the first term on the rhs is in $\mathscr{B}\big(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\big)$. The second and the last one are in $\mathscr{B}\big(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\big)$ by lemma 2.11(ii) and the boundedness of $\langle q^1\rangle^\vartheta_{,1}$, respectively. Moreover,

$$\langle q^1\rangle^\vartheta R(z)q^1 V_{,1}R(z) = R(z)\langle q^1\rangle^\vartheta q^1 V_{,1}R(z) + [\langle q^1\rangle^\vartheta, R(z)]q^1 V_{,1}R(z)$$

is in $\mathscr{B}\big(\mathcal{H}(\Omega), \mathcal{H}_0^1(\Omega)\big)$ by assumption 2.2.3 and lemma 2.11(i). Thus, all the terms in the expression of B are at least of class $C^\vartheta(A)$. This implies the claim. □

2.3.4. The main result

Proposition 2.15. $\forall\lambda \in \mathbb{R}\backslash\mathcal{T}$, $\widetilde{\varrho}_H^A(\lambda) > 0$.

Proof. Corollary 2.10 and proposition 2.14 imply that both H_0 and H are of class $C_u^1(A)$. Furthermore, $R(i) - R_0(i)$ is compact by proposition 2.2, with the result that $\widetilde{\varrho}_H^A = \widetilde{\varrho}_{H_0}^A$ due to [2, theorem 7.2.9]. Finally, since $\widetilde{\varrho}_{H_0}^A \geqslant \varrho_{H_0}^A$ [2, proposition 7.2.6], we can conclude using corollary 2.10. □

Summing up, we result in the following spectral properties of H.

Theorem 2.16. *Let ω be a bounded open connected set in $\mathbb{R}^{d-1}$, $d \geqslant 2$, and denote by $\mathcal{T}$ the set of eigenvalues of $-\Delta_D^\omega$. Let H be the operator* (1.3) *with $\Omega := \mathbb{R} \times \omega$, subject to Dirichlet boundary conditions, and satisfying assumptions* 2.1 *and* 2.2. *Then*

(i) $\sigma_{\text{ess}}(H) = [\kappa, \infty)$, *where* $\kappa := \inf\mathcal{T}$,
(ii) $\sigma_{\text{sc}}(H) = \varnothing$,
(iii) $\sigma_{\text{p}}(H) \cup \mathcal{T}$ *is closed and countable,*
(iv) $\sigma_{\text{p}}(H)\backslash\mathcal{T}$ *is composed of finitely degenerated eigenvalues, which can accumulate at the points of* $\mathcal{T}$ *only.*

Proof. The claim (i) is included in proposition 2.2. Since A is conjugate to H on $\mathbb{R}\backslash\mathcal{T}$ by proposition 2.15, the assertions (ii)–(iv) follow by the abstract conjugate operator method [2, theorem 7.4.2]. □

To conclude this section, let us remark that assumptions 2.1.3 and 2.2.3 could be weakened. Firstly, we recall that the situation with $V = 0$ and $G = \rho 1$, ρ being a real-valued function greater than a strictly positive constant, is investigated in [3, 10] where the authors admit local singularities of ρ. More specifically, one assumes that $\rho = \rho_s + \rho_\ell$, where ρ_ℓ is the part satisfying a condition analogous to assumption 2.1.3, while ρ_s need not be differentiable. (In [3], supp(ρ_s) is assumed to be compact. The result of [10] is better in the sense that ρ_s is only supposed to be a short-range perturbation there. However, this requires strengthening of the condition analogous to assumption 2.1.2 about the decay of ρ at infinity.) Secondly, the optimal conditions one has to impose on the potential of a Schrödinger operator are known [6, 2].

3. Curved tubes

In this part, we use theorem 2.16 in order to find geometric sufficient conditions which guarantee that the spectral results of the theorem hold true for curved tubes.

3.1. Geometric preliminaries

3.1.1. The reference curve. Given $d \geqslant 2$, let $p : \mathbb{R} \to \mathbb{R}^d$ be a regular unit-speed smooth (i.e. C^∞-smooth) curve satisfying the following hypothesis.

Assumption 3.1. *There exists a collection of d smooth mappings $e_i : \mathbb{R} \to \mathbb{R}^d$ with the following properties:*

1. $\forall i, j \in \{1, \ldots, d\}, \forall s \in \mathbb{R}, e_i(s) \cdot e_j(s) = \delta_{ij}$,
2. $\forall i \in \{1, \ldots, d-1\}, \forall s \in \mathbb{R}$, *the* ith *derivative* $p^{(i)}(s)$ *of* $p(s)$ *lies in the span of* $e_1(s), \ldots, e_i(s)$,

3. $e_1 = \dot{p}$,
4. $\forall s \in \mathbb{R}$, $\{e_1(s), \ldots, e_d(s)\}$ *has the positive orientation,*
5. $\forall i \in \{1, \ldots, d-1\}$, $\forall s \in \mathbb{R}$, $\dot{e}_i(s)$ *lies in the span of* $e_1(s), \ldots, e_{i+1}(s)$.

Here and in the following, '$\cdot$' denotes the inner product in $\mathbb{R}^d$.

Remark 3.1. A vector field with the property 1 is called a *moving frame* along p and it is a *Frenet frame* if it satisfies 2 in addition, cf [22, section 1.2]. A sufficient condition to ensure the existence of the frame of assumption 3.1 is to require that [22, proposition 1.2.2], for all $s \in \mathbb{R}$, the vectors $\dot{p}(s), p^{(2)}(s), \ldots, p^{(d-1)}(s)$ are linearly independent. This is always satisfied if $d = 2$. However, we do not assume *a priori* the above non-degeneracy condition for $d \geqslant 3$ because it excludes the curves such that, for some open $I \subseteq \mathbb{R}$, $p \upharpoonright I$ lies in a lower-dimensional subspace of $\mathbb{R}^d$.

The properties of $\{e_1, \ldots, e_d\}$ summarized in assumption 3.1 yield [22, section 1.3] the Serret–Frenet formulae

$$\dot{e}_i = \mathcal{K}_i{}^j e_j \tag{3.1}$$

with $\mathcal{K} \equiv \left(\mathcal{K}_i{}^j\right)$ being a skew-symmetric $d \times d$ matrix defined by

$$\mathcal{K} := \begin{pmatrix} 0 & \kappa_1 & & 0 \\ -\kappa_1 & \ddots & \ddots & \\ & \ddots & \ddots & \kappa_{d-1} \\ 0 & & -\kappa_{d-1} & 0 \end{pmatrix}. \tag{3.2}$$

Here κ_i is called the ith *curvature* of p. Under our assumption 3.1, the curvatures are smooth functions of the arc-length parameter $s \in \mathbb{R}$.

3.1.2. The appropriate moving frame. In this subsection, we introduce another moving frame along p, which better reflects the geometry of the curve, and will be used later to define a tube about it. We shall refer to it as the *Tang frame* because it is a natural generalization of the Tang frame known from the theory of three-dimensional waveguides [31, 17, 11]. Our construction follows the generalization introduced in [8].

Let the $(d-1) \times (d-1)$ matrix $\left(\mathcal{R}_\mu{}^\nu\right)$ be defined by the system of differential equations

$$\dot{\mathcal{R}}_\mu{}^\nu + \mathcal{R}_\mu{}^\alpha \mathcal{K}_\alpha{}^\nu = 0 \tag{3.3}$$

with $\left(\mathcal{R}_\mu{}^\nu(s_0)\right)$ being a rotation matrix in $\mathbb{R}^{d-1}$ for some $s_0 \in \mathbb{R}$ as initial condition, i.e.

$$\det\left(\mathcal{R}_\mu{}^\nu(s_0)\right) = 1 \qquad \text{and} \qquad \delta_{\alpha\beta}\mathcal{R}_\mu{}^\alpha(s_0)\mathcal{R}_\nu{}^\beta(s_0) = \delta_{\mu\nu}. \tag{3.4}$$

The solution of (3.3) exists and is smooth by standard arguments in the theory of differential equations (cf [25, section 4]). Furthermore, conditions (3.4) are satisfied for *all* $s_0 \in \mathbb{R}$. Indeed, by means of Liouville's formula [25, theorem 4.7.1] and $\operatorname{tr}(\mathcal{K}) = 0$, one checks that $\det\left(\mathcal{R}_\mu{}^\nu\right) = 1$ identically, while the validity of the second condition for all $s_0 \in \mathbb{R}$ is obtained via the skew-symmetry of $\mathcal{K}$:

$$\left(\delta_{\alpha\beta}\mathcal{R}_\mu{}^\alpha\mathcal{R}_\nu{}^\beta\right)^{\cdot} = -\mathcal{R}_\mu{}^\alpha\left(\delta_{\gamma\beta}\mathcal{K}_\alpha{}^\gamma + \delta_{\alpha\gamma}\mathcal{K}_\beta{}^\gamma\right)\mathcal{R}_\nu{}^\beta = 0.$$

We set

$$\mathcal{R} \equiv \left(\mathcal{R}_i{}^j\right) := \begin{pmatrix} 1 & 0 \\ 0 & \left(\mathcal{R}_\mu{}^\nu\right) \end{pmatrix}$$

and introduce the Tang frame as the moving frame $\{\tilde{e}_1, \ldots, \tilde{e}_d\}$ along p defined by

$$\tilde{e}_i := \mathcal{R}_i{}^j e_j. \tag{3.5}$$

Combining (3.1) with (3.3), one easily finds

$$\dot{\tilde{e}}_1 = \kappa_1 e_2 \qquad \text{and} \qquad \dot{\tilde{e}}_\mu = \mathcal{R}_\mu{}^\alpha \mathcal{K}_\alpha{}^1 e_1 = -\kappa_1 \mathcal{R}_\mu{}^2 e_1. \tag{3.6}$$

The interest of the Tang frame will appear in the following subsection.

3.1.3. The tube. Let ω be a bounded open connected set in $\mathbb{R}^{d-1}$. Without loss of generality, we assume that ω is translated so that its centre of mass is at the origin. Let $\Omega := \mathbb{R} \times \omega$ be a straight tube. We define the curved tube Γ of the same cross-section ω about p as the image of the mapping

$$\mathscr{L} : \Omega \to \mathbb{R}^d \qquad (s, u^2, \ldots, u^d) \mapsto p(s) + \tilde{e}_\mu(s) u^\mu \tag{3.7}$$

i.e. $\Gamma := \mathscr{L}(\Omega)$.

As already mentioned in the introduction, the shape of the curved tube Γ of cross-section ω about p depends on the choice of rotations $(\mathcal{R}_\mu{}^\nu)$ in (3.5), unless ω is rotation invariant. As usual in the theory of quantum waveguides (see, e.g. [11, 8]), we restrict ourselves to the technically most advantageous choice determined by (3.3), i.e. when the cross-section ω rotates along p w.r.t. the Tang frame (another choice can be found in [14]).

We write $u \equiv (u^2, \ldots, u^d)$, define $a := \sup_{u \in \omega} |u|$ and always assume.

Assumption 3.2.

1. *$\kappa_1 \in L^\infty(\mathbb{R})$ and $a\|\kappa_1\|_\infty < 1$,*
2. *Γ does not overlap itself.*

Then, the mapping $\mathscr{L} : \Omega \to \Gamma$ is a diffeomorphism. Indeed, by virtue of the inverse function theorem, the first condition guarantees that it is a local diffeomorphism which is global through the injectivity induced by the second condition. Consequently, $\mathscr{L}^{-1}$ determines a system of global (*geodesic* or *Fermi*) 'coordinates' (s, u). At the same time, the tube Γ can be identified with the Riemannian manifold (Ω, g), where $g \equiv (g_{ij})$ is the metric tensor induced by the immersion (3.7), that is $g_{ij} := \mathscr{L}_{,i} \cdot \mathscr{L}_{,j}$. Formulae (3.6) yield

$$g = \operatorname{diag}(h^2, 1, \ldots, 1) \qquad \text{with} \quad h(s, u) := 1 + u^\mu \mathcal{R}_\mu{}^\alpha(s) \mathcal{K}_\alpha{}^1(s). \tag{3.8}$$

Note that the metric tensor (3.8) is diagonal due to our special choice of the 'transverse' frame $\{\tilde{e}_2, \ldots, \tilde{e}_d\}$, which is the advantage of the Tang frame.

We set $|g| := \det(g) = h^2$, which defines through $\mathrm{d}v := h(s, u)\, \mathrm{d}s\, \mathrm{d}u$ the volume element of Γ; here $\mathrm{d}u$ denotes the $(d-1)$-dimensional Lebesgue measure in ω.

Remark 3.2 (Low-dimensional examples)**.** When $d = 2$, the cross-section ω is just the interval $(-a, a)$, the curve p has only one curvature $\kappa := \kappa_1$, the rotation matrix $(\mathcal{R}_\mu{}^\nu)$ equals (the scalar) 1 and

$$h(s, u) = 1 - \kappa(s) u.$$

If $d = 3$, it is convenient to make the ansatz

$$(\mathcal{R}_\mu{}^\nu) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

where α is a real-valued differentiable function. Then, it is easy to see that (3.3) reduces to the differential equation $\dot{\alpha} = \tau$, where τ is the torsion of p, i.e. one puts $\kappa := \kappa_1$ and $\tau := \kappa_2$. Choosing α as an integral of τ, we can write

$$h(s, u) = 1 - \kappa(s)[u^2 \cos\alpha(s) + u^3 \sin\alpha(s)].$$

Remark 3.3 (On assumption 3.2)**.** If p were a compact embedded curve, then assumption 3.2 could always be achieved for sufficiently small a. In general, however, one cannot exclude self-intersections of the tube using the local geometry of an embedded curve p only. One way to avoid this disadvantage would be to consider (Ω, g) as an abstract Riemannian manifold where only the curve p is embedded in $\mathbb{R}^d$. Nonetheless, in the present paper, we prefer to assume assumption 3.2.2 *a priori* because Γ does not have a physical meaning if it is self-intersecting. Finding global geometric conditions on p ensuring the validity of assumption 3.2.2 is an interesting question, which is beyond the scope of the present paper, however.

3.2. The Laplacian

Our object of interest is the Dirichlet Laplacian (1.2), with Γ defined by (3.7). We construct it as follows. Using the diffeomorphism (3.7), we identify the Hilbert space $L^2(\Gamma)$ with $L^2(\Omega, \mathrm{d}v)$ and consider on the latter the Dirichlet form

$$\widetilde{Q}(\varphi, \psi) := \int_\Omega \overline{\varphi_{,i}}\, g^{ij} \psi_{,j}\, \mathrm{d}v \qquad \varphi, \psi \in \mathcal{D}(\widetilde{Q}) := \mathcal{H}_0^1(\Omega, \mathrm{d}v) \tag{3.9}$$

where $(g^{ij}) := g^{-1}$. The form $\widetilde{Q}$ is clearly densely defined, non-negative, symmetric and closed on its domain. Consequently, there exists a unique non-negative self-adjoint operator $\widetilde{H}$ satisfying $\mathcal{D}(\widetilde{H}) \subset \mathcal{D}(\widetilde{Q})$ associated with $\widetilde{Q}$. We have

$$\widetilde{H}\psi = -|g|^{-1/2}\partial_i |g|^{1/2} g^{ij} \partial_j \psi \tag{3.10}$$

$$\psi \in \mathcal{D}(\widetilde{H}) = \left\{\psi \in \mathcal{H}_0^1(\Omega, \mathrm{d}v) : \partial_i |g|^{1/2} g^{ij} \partial_j \psi \in L^2(\Omega, \mathrm{d}v)\right\}. \tag{3.11}$$

That is, $\widetilde{H}$ is the Laplacian (1.2) expressed in the coordinates (s, u).

In order to apply theorem 2.16, we transform $\widetilde{H}$ into a unitarily equivalent operator H of the form (1.3) acting on the Hilbert space $\mathcal{H}(\Omega) := L^2(\Omega)$, without the additional weight $|g|^{1/2}$ in the volume element. This is achieved by means of the unitary mapping $\mathcal{U} : L^2(\Omega, \mathrm{d}v) \to \mathcal{H}(\Omega)$, $\psi \mapsto |g|^{1/4}\psi$. Defining $H := \mathcal{U}\widetilde{H}\mathcal{U}^{-1}$, one has

$$H\psi = -|g|^{-1/4}\partial_i |g|^{1/2} g^{ij} \partial_j |g|^{-1/4} \psi, \tag{3.12}$$

$$\psi \in \mathcal{D}(H) = \left\{\psi \in \mathcal{H}_0^1(\Omega) : \partial_i |g|^{1/2} g^{ij} \partial_j |g|^{-1/4} \psi \in L^2(\Omega)\right\}. \tag{3.13}$$

Commuting $|g|^{-1/4}$ with the gradient components in the expression for H, we obtain on $\mathcal{D}(H)$

$$H = -\partial_i g^{ij} \partial_j + V \tag{3.14}$$

where

$$V := -\frac{5}{4}\frac{(h_{,1})^2}{h^4} + \frac{1}{2}\frac{h_{,11}}{h^3} - \frac{1}{4}\frac{\delta^{\mu\nu} h_{,\mu} h_{,\nu}}{h^2} + \frac{1}{2}\frac{\delta^{\mu\nu} h_{,\mu\nu}}{h}. \tag{3.15}$$

Actually, (3.14) with (3.15) is a general formula valid for any smooth metric of the form $g = \mathrm{diag}(h^2, 1, \ldots, 1)$. In our special case with h given by (3.8), we find that $h_{,\mu\nu} = 0$, $\delta^{\mu\nu} h_{,\mu} h_{,\nu} = \delta^{\alpha\beta} \mathcal{K}_\alpha{}^1 \mathcal{K}_\beta{}^1$ by (3.4), while (3.3) gives

$$\begin{aligned} h_{,1}(\cdot, u) &= u^\mu \mathcal{R}_\mu{}^\alpha \left(\dot{\mathcal{K}}_\alpha{}^1 - \mathcal{K}_\alpha{}^\beta \mathcal{K}_\beta{}^1\right) \\ h_{,11}(\cdot, u) &= u^\mu \mathcal{R}_\mu{}^\alpha \left(\ddot{\mathcal{K}}_\alpha{}^1 - \dot{\mathcal{K}}_\alpha{}^\beta \mathcal{K}_\beta{}^1 - 2\mathcal{K}_\alpha{}^\beta \dot{\mathcal{K}}_\beta{}^1 + \mathcal{K}_\alpha{}^\beta \mathcal{K}_\beta{}^\gamma \mathcal{K}_\gamma{}^1\right). \end{aligned} \tag{3.16}$$

3.3. Results

It remains to impose decay conditions on the curvatures of p (and their derivatives) in order that the operator (3.14) satisfies assumptions 2.1 and 2.2.

Let us first consider the more general situation where the matrix (g^{ij}) is equal to $\mathrm{diag}(h^{-2}, 1, \ldots, 1)$ with the explicit dependence of h on s and u not specified. One shows that it is sufficient to impose the following hypotheses.

Assumption 3.3. *Uniformly for* $u \in \omega$,

1. $h(s,u) \longrightarrow 1$ *as* $|s| \to \infty$,
2. $h_{,11}(s,u)$, $(\delta^{\mu\nu} h_{,\mu} h_{,\nu})(s,u)$, $\delta^{\mu\nu} h_{,\mu\nu}(s,u) \longrightarrow 0$ *as* $|s| \to \infty$,
3. $\exists \vartheta \in (0,1]$ *s.t.*

$$h_{,1}(s,u),\, h_{,111}(s,u),\, (\delta^{\mu\nu} h_{,\mu} h_{,\nu})_{,1}(s,u),\, \delta^{\mu\nu} h_{,1\mu\nu}(s,u) = \mathcal{O}(|s|^{-(1+\vartheta)}).$$

Indeed, the first hypothesis supplies assumption 2.1.2, while assumption 2.1.1 is fulfilled due to basic assumption 3.2. Next, since h is a smooth function, assumption 3.3.2 together with the behaviour of $h_{,1}$ in assumption 3.3.3 are sufficient to ensure both assumption 2.2.1 and assumption 2.2.2. It is also clear that the asymptotic behaviour of $h_{,1}$ in assumption 3.3.3 supplies assumption 2.1.3. Assumption 2.1.4 holds true due to assumption 2.1.3 and the particular form of (g^{ij}). It remains to check assumption 2.2.3. This is easily done by calculating the derivative of the potential (3.15):

$$V_{,1} = 5\frac{(h_{,1})^3}{h^5} - 4\frac{h_{,1}h_{,11}}{h^4} + \frac{h_{,111}}{2h^3} + \frac{\delta^{\mu\nu}}{2}\left(\frac{h_{,1}h_{,\mu}h_{,\nu}}{h^3} - \frac{h_{,1}h_{,\mu\nu} + h_{,1\mu}h_{,\nu}}{h^2} + \frac{h_{,1\mu\nu}}{h}\right).$$

With h given by (3.8), we find in addition to (3.16) that $h_{,1\mu\nu} = 0$ and

$$(\delta^{\mu\nu} h_{,\mu} h_{,\nu})_{,1} = 2\delta^{\alpha\beta} \dot{\mathcal{K}}_\alpha{}^1 \mathcal{K}_\beta{}^1$$

$$h_{,111}(\cdot,u) = u^\mu \mathcal{R}_\mu{}^\alpha \big(\dddot{\mathcal{K}}_\alpha{}^1 - \ddot{\mathcal{K}}_\alpha{}^\beta \mathcal{K}_\beta{}^1 - 3\mathcal{K}_\alpha{}^\beta \ddot{\mathcal{K}}_\beta{}^1 - 3\dot{\mathcal{K}}_\alpha{}^\beta \dot{\mathcal{K}}_\beta{}^1 + \dot{\mathcal{K}}_\alpha{}^\beta \mathcal{K}_\beta{}^\gamma \mathcal{K}_\gamma{}^1$$
$$+ 2\mathcal{K}_\alpha{}^\beta \dot{\mathcal{K}}_\beta{}^\gamma \mathcal{K}_\gamma{}^1 + 3\mathcal{K}_\alpha{}^\beta \mathcal{K}_\beta{}^\gamma \dot{\mathcal{K}}_\gamma{}^1 - \mathcal{K}_\alpha{}^\beta \mathcal{K}_\beta{}^\gamma \mathcal{K}_\gamma{}^\delta \mathcal{K}_\delta{}^1\big).$$

Since $|u^\mu \mathcal{R}_\mu{}^\alpha| < a$, assumption 3.3 holds true provided we impose the following conditions on the curvatures:

Assumption 3.4.

1. $\forall \alpha \in \{2,\ldots,d\}$, $\mathcal{K}_\alpha{}^1(s), \ddot{\mathcal{K}}_\alpha{}^1(s) \longrightarrow 0$ *as* $|s| \to \infty$,
2. $\forall \alpha, \beta \in \{2,\ldots,d\}$, $\mathcal{K}_\alpha{}^\beta, \dot{\mathcal{K}}_\alpha{}^2 \in L^\infty(\mathbb{R})$,
3. $\exists \vartheta \in (0,1]$ *s.t.* $\forall \alpha \in \{2,\ldots,d\}$,

$$\dot{\mathcal{K}}_\alpha{}^1(s), \dddot{\mathcal{K}}_\alpha{}^1(s), \mathcal{K}_\alpha{}^2(s), \ddot{\mathcal{K}}_\alpha{}^2(s), \big(\dot{\mathcal{K}}_\alpha{}^\beta \mathcal{K}_\beta{}^2\big)(s), \big(\mathcal{K}_\alpha{}^\beta \dot{\mathcal{K}}_\beta{}^2\big)(s) = \mathcal{O}(|s|^{-(1+\vartheta)}).$$

Remark 3.4. These conditions reduce to those of theorem 1.1 provided $d = 2$. When $d = 3$, it is sufficient to assume the conditions of theorem 1.1 for the first curvature, and $\dot{\kappa}_2 \in L^\infty(\mathbb{R})$ and $\kappa_2(s), \ddot{\kappa}_2(s) = \mathcal{O}(|s|^{-(1+\vartheta)})$ for some $\vartheta \in (0,1]$.

We conclude this section by applying theorem 2.16.

Theorem 3.5. *Let* Γ *be a tube defined via* (3.7) *about a smooth infinite curve embedded in* $\mathbb{R}^d$. *Consider assumptions 3.1, 3.2 and 3.4. Then all the spectral results (i)–(iv) of theorem 2.16 hold true for the Dirichlet Laplacian on* $L^2(\Gamma)$.

4. Curved strips on surfaces

In this final section, we investigate the situation where the ambient space is a general Riemannian manifold instead of the Euclidean space $\mathbb{R}^d$. We restrict ourselves to $d = 2$, i.e. Γ is a strip around an infinite curve in an (abstract) two-dimensional surface. We refer to [23] for basic spectral properties of $-\Delta_D^\Gamma$ and geometric details.

4.1. Preliminaries

Let $\mathcal{A}$ be a smooth connected complete non-compact two-dimensional Riemannian manifold of bounded Gauss curvature K. Let $p : \mathbb{R} \to \mathcal{A}$ be a smooth unit-speed curve embedded in $\mathcal{A}$ with (geodesic) curvature κ and denote by $n : \mathbb{R} \to T_{p(\cdot)}\mathcal{A}$ a smooth unit normal vector field along p. Given $a > 0$, we consider the straight strip $\Omega := \mathbb{R} \times (-a, a)$ and define a curved strip Γ of same width over p as an a-tubular neighbourhood of p in $\mathcal{A}$ by

$$\Gamma := \mathscr{L}(\Omega) \qquad \text{where} \quad \mathscr{L} : (s, u) \mapsto \exp_{p(s)}(u\,n(s)). \tag{4.1}$$

Note that $s \mapsto \mathscr{L}(s, u)$ traces the curves parallel to p at a fixed distance $|u|$, while the curve $u \mapsto \mathscr{L}(s, u)$ is a unit-speed geodesic orthogonal to p for any fixed s. We always assume

Assumption 4.1. $\mathscr{L} : \Omega \to \Gamma$ *is a diffeomorphism.*

Then $\mathscr{L}^{-1}$ determines a system of Fermi 'coordinates' (s, u), i.e. the geodesic coordinates based on p. The metric tensor of Γ in these coordinates acquires [18, section 2.4] the diagonal form

$$g(s, u) = \operatorname{diag}(h^2(s, u), 1) \tag{4.2}$$

where h is a smooth function satisfying the Jacobi equation

$$h_{,22} + K h = 0 \qquad \text{with} \quad \begin{cases} h(\cdot, 0) = 1 \\ h_{,2}(\cdot, 0) = -\kappa. \end{cases} \tag{4.3}$$

Here K and κ are considered as functions of the Fermi coordinates (the sign of κ being uniquely determined up to the re-parametrization $s \mapsto -s$ or the choice of n). The determinant of the metric tensor, $|g| := \det(g) = h^2$, defines through $\mathrm{d}v := h(s, u)\,\mathrm{d}s\,\mathrm{d}u$ the area element of the strip.

Assuming that the metric g is uniformly elliptic in the sense that

Assumption 4.2. $\exists c_\pm \in (0, \infty)$ *s.t.* $\forall (s, u) \in \Omega, c_- \leqslant h(s, u) \leqslant c_+$

holds true, the Dirichlet Laplacian corresponding to Γ can be defined in the same way as in section 3.2, i.e. as the operator $\widetilde{H}$ associated with the form (3.9), satisfying (3.10). At the same time, we may introduce the unitarily equivalent operator H on $L^2(\Omega)$ given by (3.12) and satisfying (3.14) with (3.15).

Remark 4.1. If assumption 4.2 holds true, then the inverse function theorem together with (4.3) yields that assumption 4.1 is satisfied for all sufficiently small a provided the strip Γ does not overlap itself. Assumption 4.2 is satisfied, for instance, if Γ is a sufficiently thin strip on a ruled surface, cf [23, section 7].

4.2. Results

In view of the more general approach in the beginning of section 3.3, we see that assumption 3.3 (with $d = 2$) guarantees assumptions 2.1 and 2.2 also in the present case. Applying theorem 2.16, we obtain, with $\mathcal{T} = \{n^2\nu_1\}_{n=1}^{\infty}$ where $\nu_1 := \pi^2/(2a)^2$, the following result:

Theorem 4.2. *Let Γ be a tubular neighbourhood of radius $a > 0$ about a smooth infinite curve, which is embedded in a smooth connected complete non-compact surface of bounded curvature. Consider assumptions 4.1, 4.2 and 3.3. Then all the spectral results (i)–(iv) of theorem 2.16 hold true for the Dirichlet Laplacian on $L^2(\Gamma)$.*

Assume now that the strip is *flat* in the sense of [23], i.e. the curvature K is equal to zero everywhere on Γ. Then the Jacobi equation (4.3) has the explicit solution (cf (3.8) for $d = 2$)

$$h(s, u) = 1 - \kappa(s)u \tag{4.4}$$

and assumption 3.3 can be replaced by some conditions on the decay of the curvature κ at infinity, namely, we adopt assumption 3.4 with $\kappa_1 \equiv \kappa$ and $\mathcal{K}_\mu{}^\nu = 0$ (cf the assumptions of theorem 1.1). At the same time, it is easy to see that assumption 4.1 and 4.2 are satisfied if assumption 3.2 holds true.

Theorem 4.3 (Flat strips)**.** *Let Γ be a tubular neighbourhood of radius $a > 0$ about a smooth infinite curve of curvature κ, which is embedded in a smooth connected complete non-compact surface of bounded curvature K such that $K \restriction \Gamma = 0$. Consider assumption 3.2 and*

(1) $\kappa(s), \ddot{\kappa}(s) \longrightarrow 0$ as $|s| \to \infty$,
(2) $\exists\vartheta \in (0, 1]$ s.t. $\dot{\kappa}(s), \dddot{\kappa}(s) = \mathcal{O}(|s|^{-(1+\vartheta)})$.

Then, all the spectral results (i)–(iv) of theorem 2.16 hold true for the Dirichlet Laplacian on $L^2(\Gamma)$.

Acknowledgments

One of the authors (DK) would like to thank M Măntoiu and R Purice for proposing to use the conjugate operator method in the theory of quantum waveguides, and for many discussions on the topic. We are very grateful to W Amrein for his careful reading of the present paper and S Richard for his useful suggestions and comments. This work was partially supported by FCT/POCTI/FEDER, Portugal, GA AS CR grant IAA 1048101 and the Fonds National Suisse.

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